# A De Bruijn – Erdős Theorem in Graphs?

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**Montreal, Quebec, Canada**

**and**

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**Charles University**
**Praha, Czech Republic**

**If *n* points in the plane ($n$ > 1) do not lie on a single line, then they determine at least *n* distinct lines**

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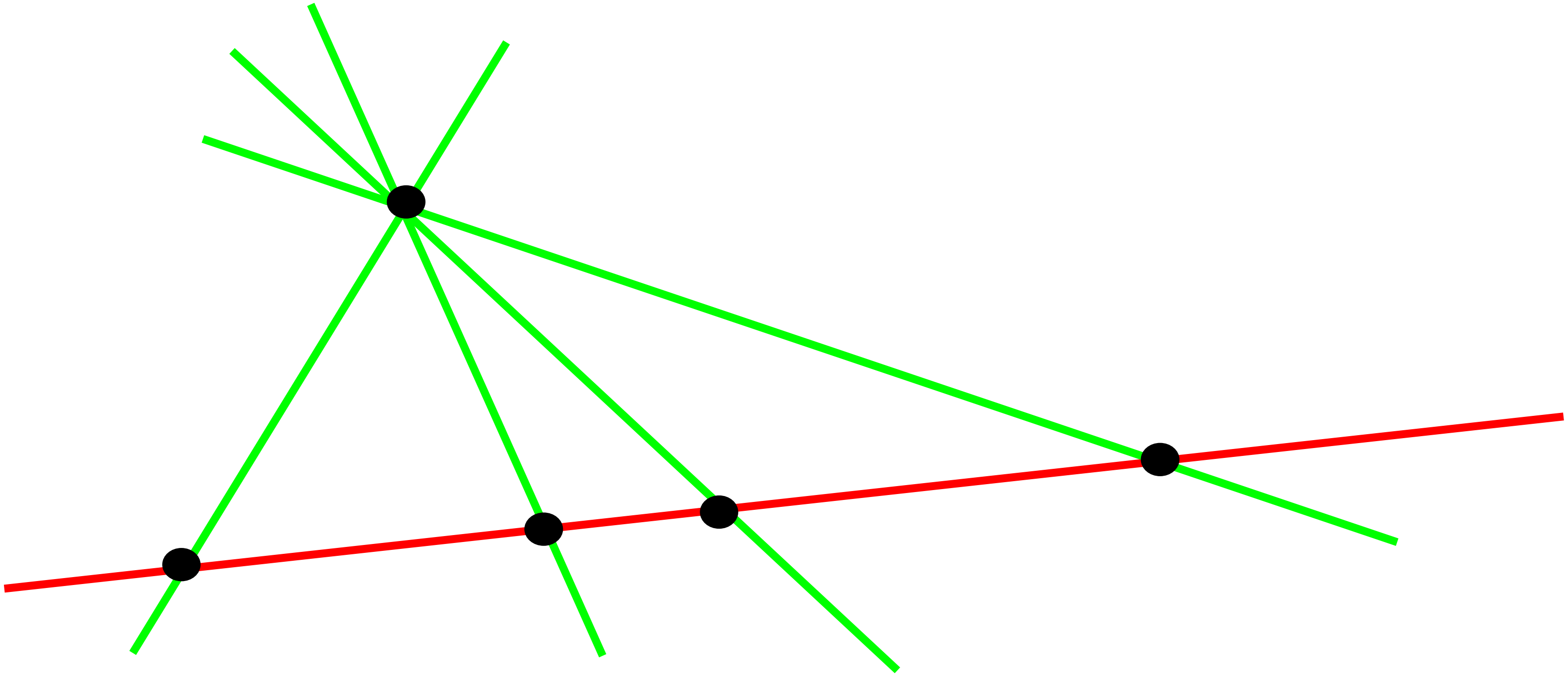

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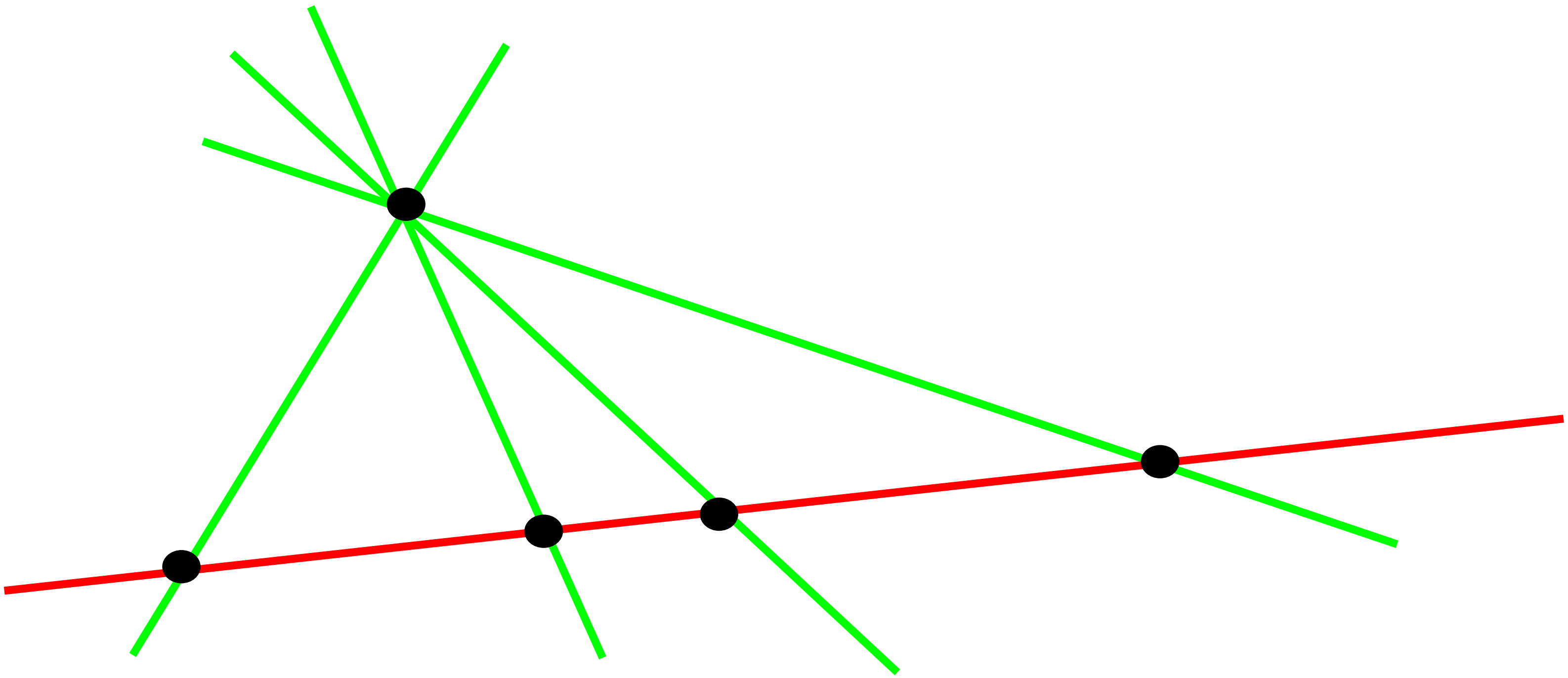

***near-pencil***

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**but often referred to (incorrectly, of course) as a "de Bruijn-Erdős Theorem"**

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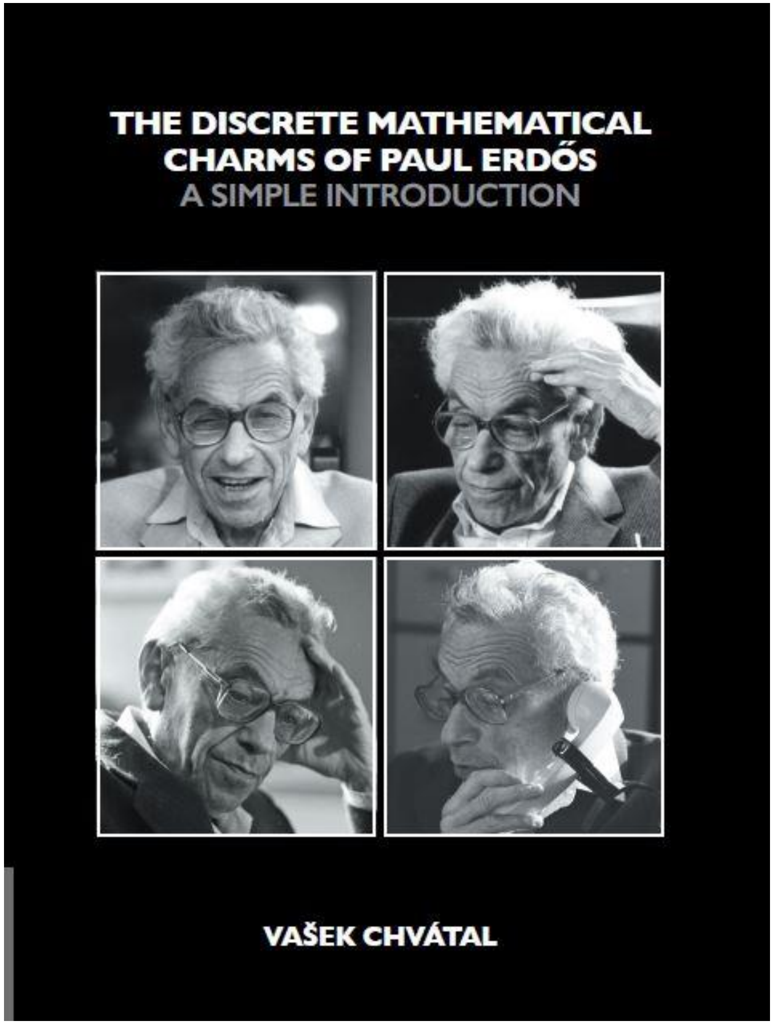
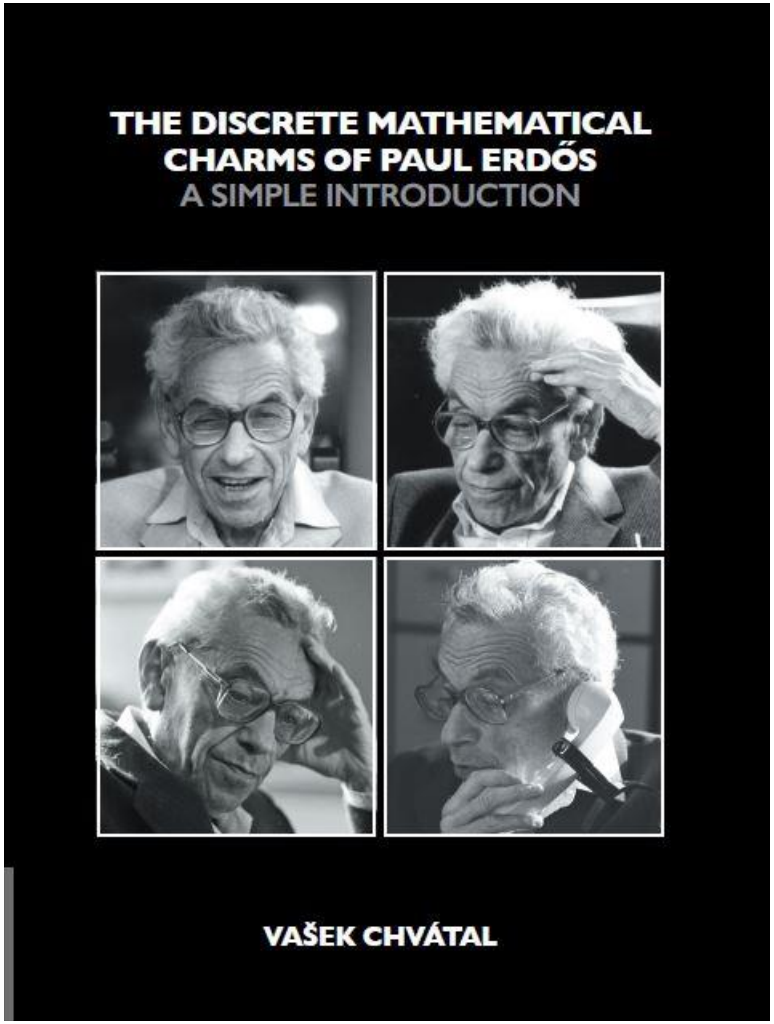
**but often referred to (incorrectly, of course) as a "de Bruijn-Erdős Theorem"**

**because it is a special case of a (far more general) theorem in**

On a combinatorial problem, *Indag. Math.* **10** (1948), 421--423




Cambridge University Press, August 2021



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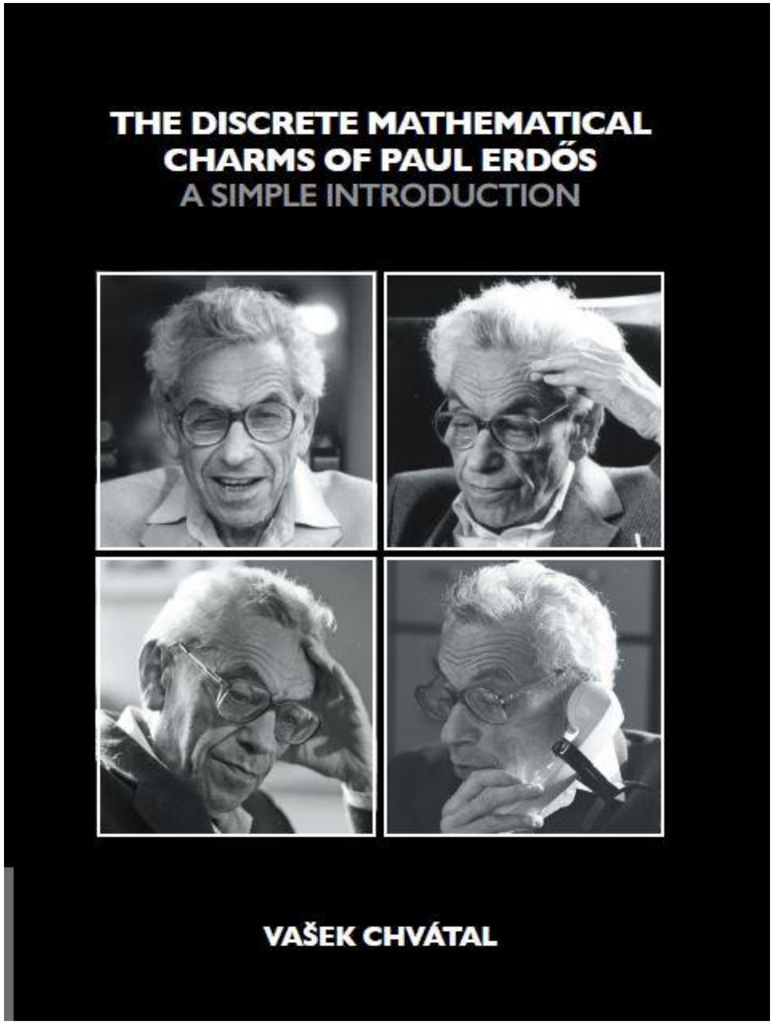




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**What other icebergs**

**could this theorem be a tip of?**

If $n$ points in the plane ($n > 1$) do not lie on a single line, then they determine at least $n$ distinct lines

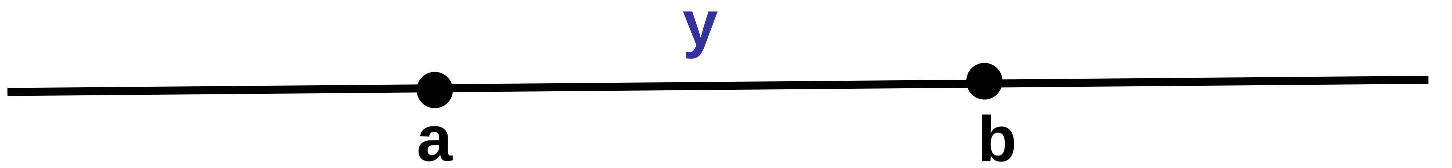
This theorem is just the tip of an iceberg, which is the de Bruijn – Erdős theorem

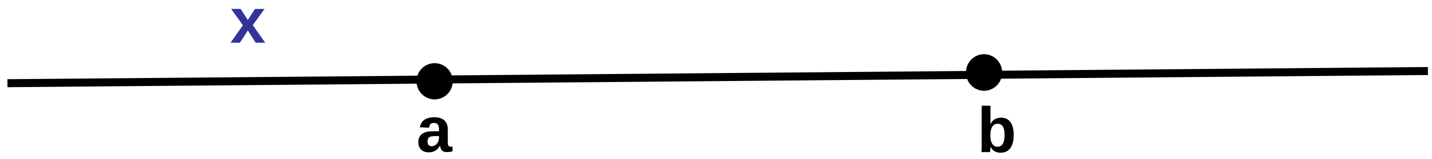
# What other icebergs could this theorem be a tip of?

**Question (Xiaomin Chen and V.C. 2006):**

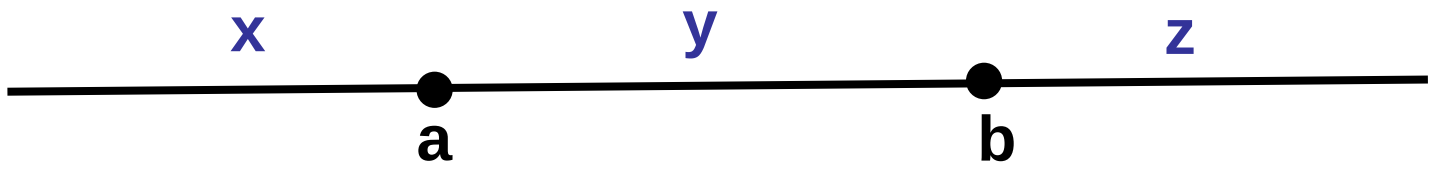
True or false? In every metric space on $n$ points ($n > 1$), there are at least $n$ distinct lines or else some line consists of all these $n$ points.

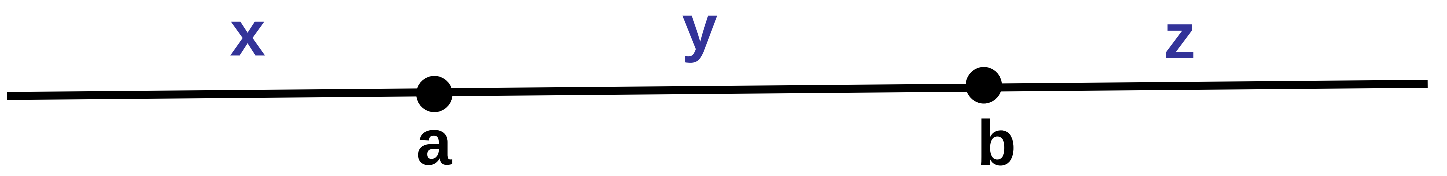

a
b


y
a
b


x
a
b

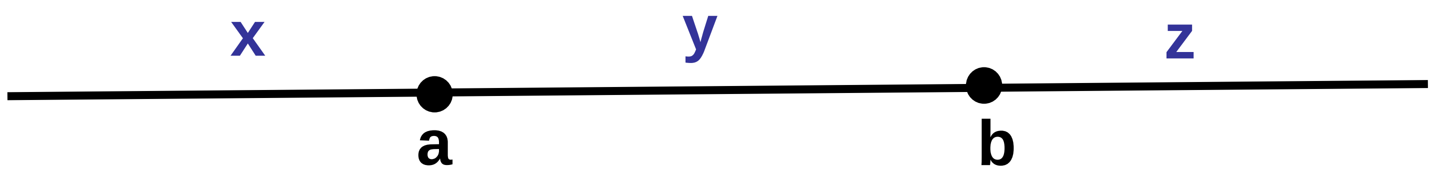

z
a
b


x
y
z
a
b




**Observation**

Line *ab* consists of
all points *x* such that *dist(x,a)+dist(a,b)=dist(x,b),*
all points *y* such that *dist(a,y)+dist(y,b)=dist(a,b),*
all points *z* such that *dist(a,b)+dist(b,z)=dist(a,z).*

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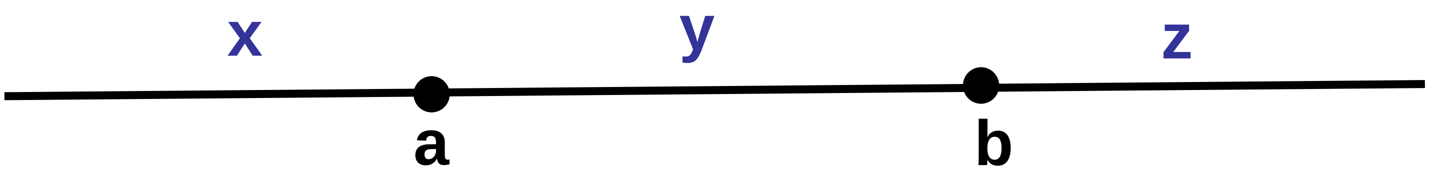
**This can be taken for a definition of a line L(*ab*)**

**in an arbitrary metric space**

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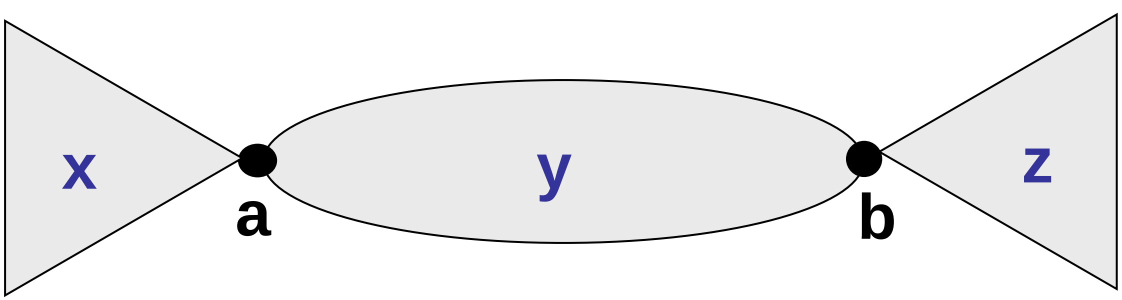

# Lines in graphs can be exotic

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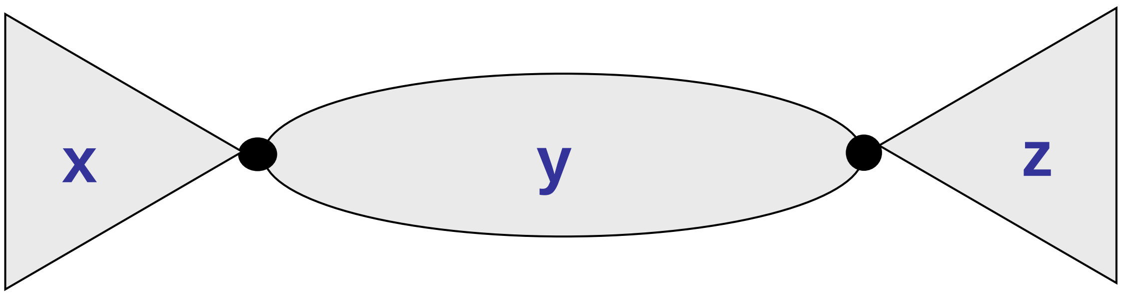
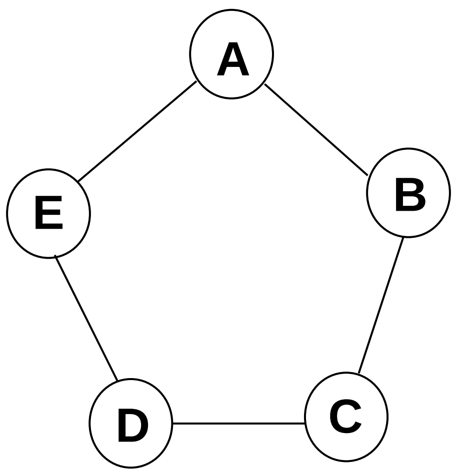
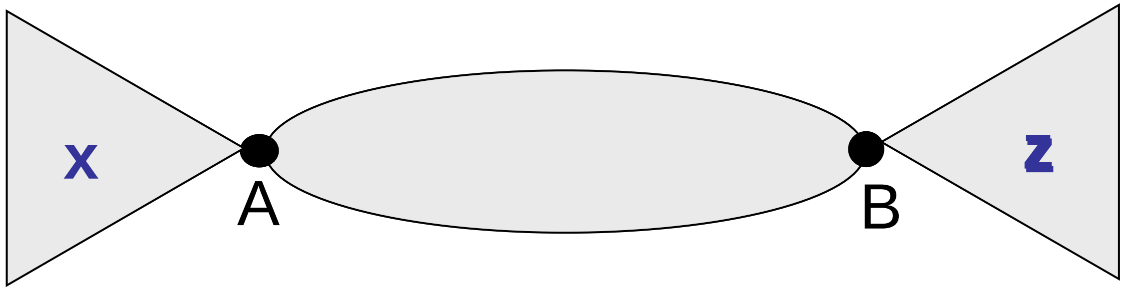
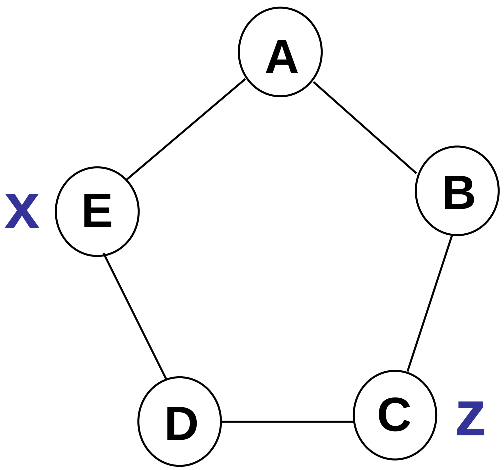
# One line can hide another!

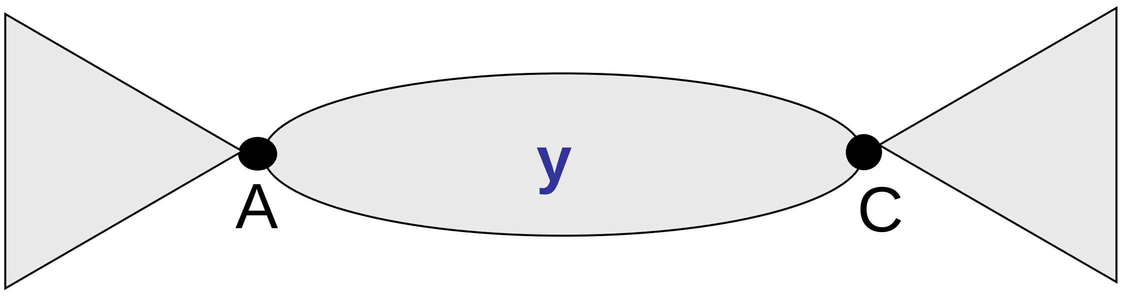

x
y
z

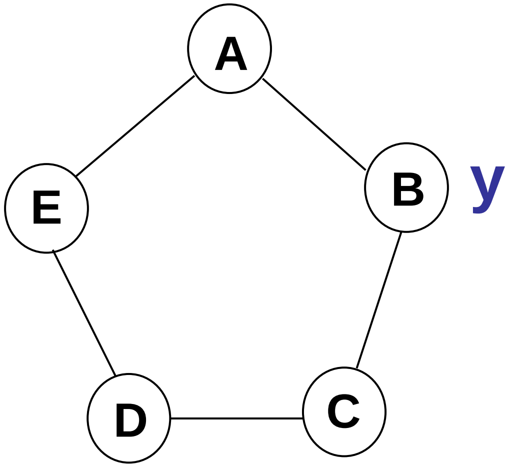

A
B
C
D
E



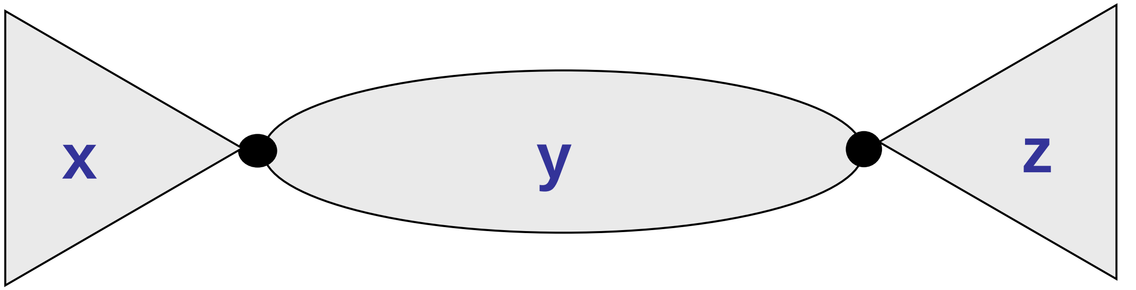



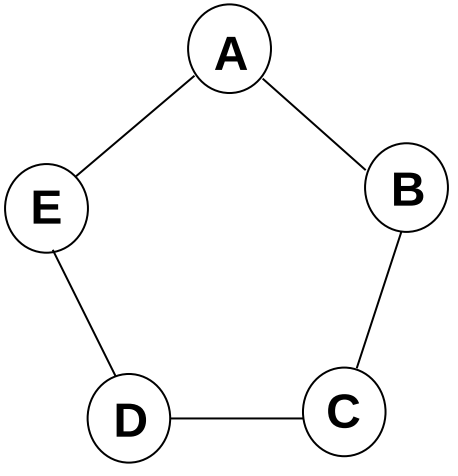

L(AB) = {E,A,B,C}




L(AC) = {A,B,C}

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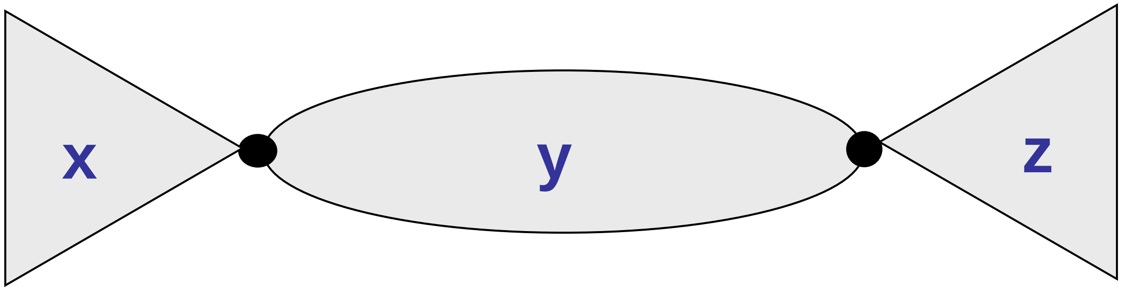


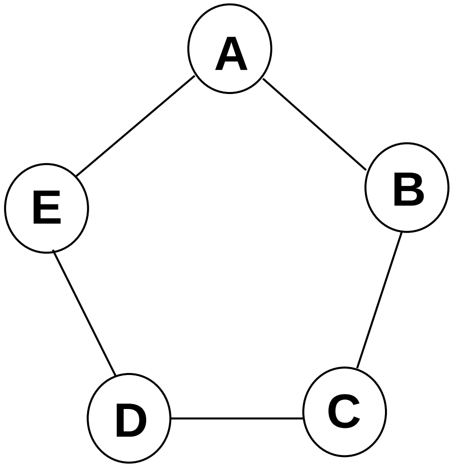


L(AB) = {E,A,B,C}

L(AC) = {A,B,C}

**One line can hide another!**

**Theorem (Xiaomin Chen, Guangda Huzhang, Peihan Miao, Kuan Yang 2015 ):**

In almost all graphs, no line is a proper superset of another.

True or false? In every metric space on $n$ points ($n > 1$), there are at least $n$ distinct lines or else some line consists of all these $n$ points.

True or false? In every metric space on *n* points ($n > 1$),
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**Theorem (Pierre Aboulker, Xiaomin Chen, Guangda Huzhang, Rohan Kapadia, Cathryn Supko 2014 ):**

In all connected graphs on $n$ vertices ($n > 1$), there are $\Omega(n^{4/7})$ distinct lines or else some line consists of all $n$ vertices.

True or false? In all connected graphs on *n* vertices (*n* > 1), there are at least *n* distinct lines or else some line consists of all these *n* vertices.

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True in all *bisplit* graphs.

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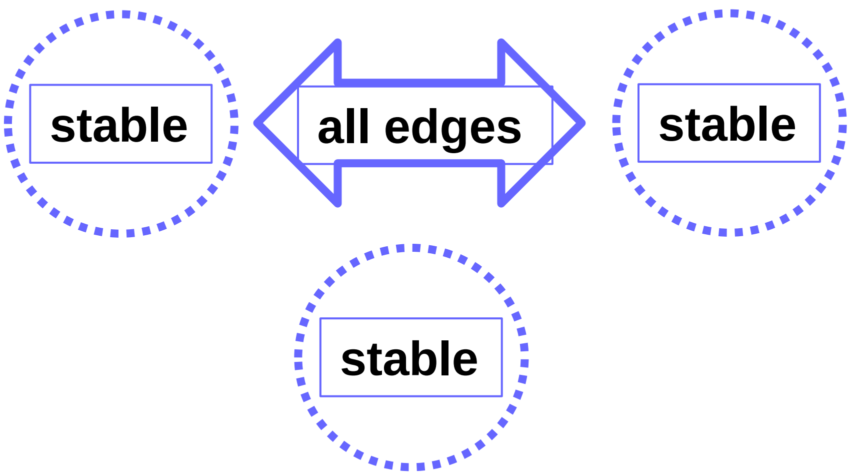

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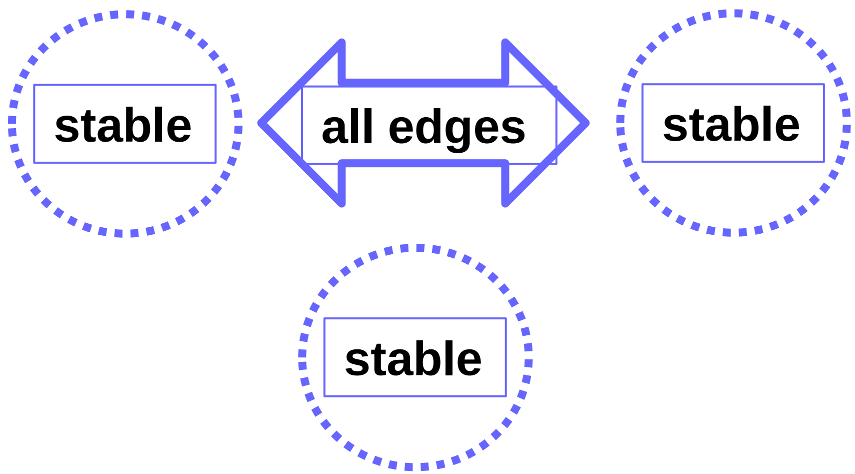


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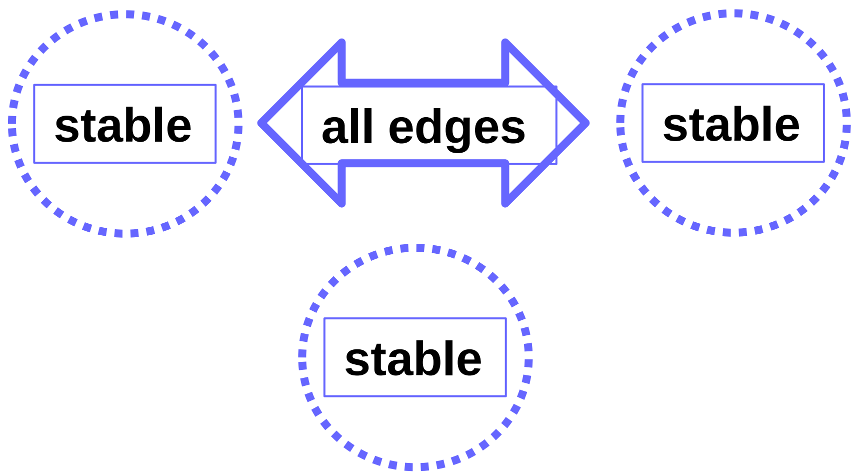


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True in all (house, hole)-free graphs.

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True in all hole-free (a.k.a. *weakly triangulated*) graphs?

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True in all (house, $C_5$, $P_5$)-free graphs?

True or false? In all connected graphs on $n$ vertices ($n > 1$), where no line consists of all $n$ vertices, there are at least $n$ distinct lines.

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Another partial answer (an easy exercise):
True in all complete multipartite graphs.

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In complete multipartite graphs on $n$ vertices ($n > 1$),
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True or false? In all connected graphs on $n$ vertices ($n > 1$),
complete multipartite graphs minimize the number of distinct lines.

True or false? In all connected graphs on $n$ vertices ($n > 1$), where no line consists of all $n$ vertices, there are at least $n$ distinct lines.

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True or false? In all connected graphs on $n$ vertices ($n > 1$), complete multipartite graphs minimize the number of distinct lines.

If the answer to this last question happens to be ‘true’, then the conjectured lower bound $n$ on the number of distinct lines (inherited from the plane geometry theorem of Erdős) is misleading and can be strengthened in the context of graphs!

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**Yori Zwols: True for** $n \leq 12$.

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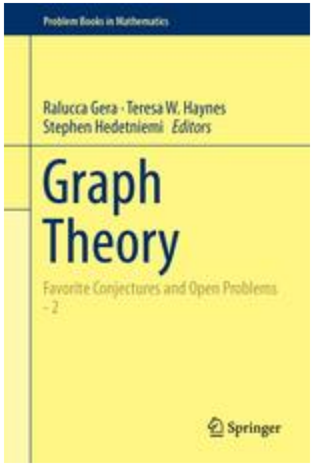
**Yori Zwols: True for** $n \leq 12$.

Each of $K(3,3,4)$, $K(1,3,3,3)$ and the complement of the Petersen graph has *15* lines.




Graph Theory
Favorite Conjectures and Open Problems - 2
Editors: **Gera**, Ralucca, **Haynes**, Teresa W., **Hedetniemi**, Stephen (Eds.)

Chapter 13:
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